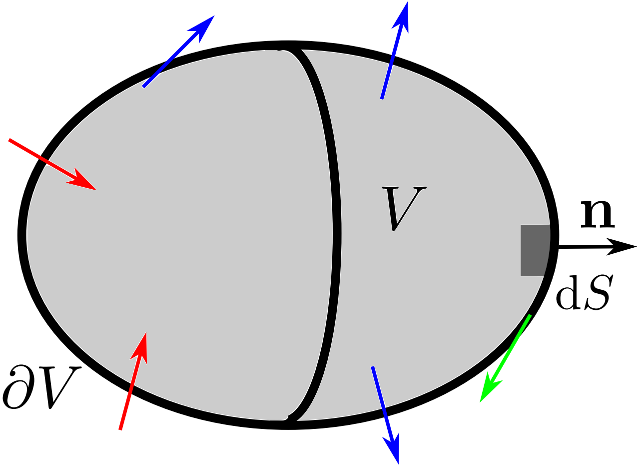
Balance equations II

Space–time dependent equations

- For certain situations time dependence is not sufficient (e. g. weather forecast).
- Necessity to add spatial dependence.
- The balance is now done in a volume $V \in \mathbb{R}^d$ and must not only take into account the destruction/creation processes but also the fluxes through the surface of V, noted ∂V.

The flux through of a quantity is the amount of s crossing ∂V per unit time.

Control volume



The continuity equation

- Let us define $\rho(t, \boldsymbol{x})$ the volume density of the property one wants to study (mass, electric charge, ...)

$$s(t) = \int_V \rho(t, \boldsymbol{x}) \mathrm{d}V.$$

- Balance equation for $s(t)$ in V reads

$$\dot{s} = -\underbrace{\text{flux}}_{=\boldsymbol{j}} \text{ through surface} + \underbrace{\text{volumic destruction/creation rate}}_{=\Sigma}.$$

The continuity equation

By defining $\Sigma = \int_V \sigma \mathrm{d}V$

$$\boxed{\dot{s} + \oint_{\partial V} \boldsymbol{j} \cdot \boldsymbol{n} \mathrm{d}S = \Sigma}$$

By using the Gauss–Ostrogradski theorem

$$\int_V \frac{\partial \rho}{\partial t} \mathrm{d}V + \int_V \boldsymbol{\nabla} \cdot \boldsymbol{j} \mathrm{d}V = \int_V \sigma \mathrm{d}V,$$

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = \sigma}$$

Example: The diffusion equation, $\rho \to C$

Fick's law $\boldsymbol{j} = -D\boldsymbol{\nabla} C$, and $\sigma = 0$

$$\frac{\partial C}{\partial t} = D\boldsymbol{\nabla}^2 C,$$

where D is the diffusion coefficient.

Example: The Navier–Stokes equations, $\rho \to \rho \boldsymbol{u}$

Momentum flux $\boldsymbol{j} = \rho \boldsymbol{u}\boldsymbol{u} + p\boldsymbol{I} - \boldsymbol{\tau}$, and $\sigma = 0$

$$\frac{\partial \rho \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}\boldsymbol{u}) = -\boldsymbol{\nabla} p + \boldsymbol{\nabla} \cdot \boldsymbol{\tau}.$$

For a Newtonian incompressible fluid

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \boldsymbol{u},$$

with ν the kinematic viscosity.