A) Fonction de corrélation :

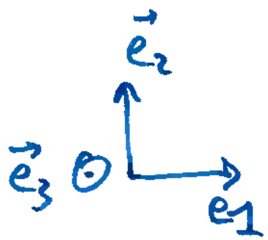
B). Fonctions de structure d'ordre 2.

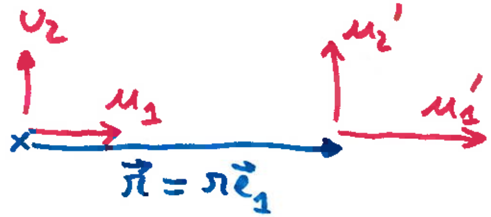
$$D_{ij}(\vec{x}, \vec{r}, t) = \overline{[u_i(\vec{x}+\vec{r}) - u_i(\vec{x})][u_j(\vec{x}+\vec{r}) - u_j(\vec{x})]}.$$

- Stat. Stationnaire.
- Homogène.
- Isotrope $\quad D_{ij}(r = |\vec{r}|)$

Propriétés.

$$\boxed{D_{ij}(\vec{r}) = a(r)\,\delta_{ij} + b(r)\,\frac{r_i r_j}{r^2}}$$





$$D_{11} = D_{LL} = a(r)\,\delta_{11} + b\,\frac{r_1 r_1}{r^2}$$
$$= a(r) + b(r)$$

$$D_{22} = D_{33} = D_{NN} = a(r)\,\delta_{22} + \cancel{b\,\frac{r_2 r_2}{r^2}}$$

$$b = D_{LL} - D_{NN}.$$

$$\boxed{D_{ij}(\vec{r}) = D_{NN}(r)\,\delta_{ij} + (D_{LL} - D_{NN})\,r_i r_j / r^2}$$

$$D_{ii} = 3D_{NN} + (D_{LL} - D_{NN}) = D_{LL} + 2D_{NN}.$$

Incompressibilité $\quad \partial_i v_i = 0 \qquad \boxed{\frac{\partial}{\partial r_i} D_{ij} = 0.}$

$$\boxed{\frac{\partial}{\partial r_i} f(r) = \frac{r_i}{r}\frac{\partial f}{\partial r}} \qquad \boxed{\frac{\partial}{\partial r_i} r_j = \delta_{ij}} \qquad \boxed{\frac{\partial}{\partial r_i} r_i = \delta_{ii} = 3.}$$

$$\frac{\partial}{\partial r_i} D_{ij} = \frac{\partial}{\partial r_j} D_{NN} + \frac{\partial}{\partial r_i}\left(D_{LL} - D_{NN}\right)\frac{r_i r_j}{r^2}$$

$$+ (D_{LL} - D_{NN})\left[\frac{3r_j}{r^2} + \frac{r_j}{r^2} + r_i r_j \frac{\partial}{\partial r_i}\left(\frac{1}{r^2}\right)\right]$$

$$= \frac{r_j}{r}\frac{\partial D_{NN}}{\partial r} + \frac{r_i}{r}\frac{\partial}{\partial r}\left(D_{LL} - D_{NN}\right)\frac{r_i r_j}{r^2}$$

$$+ (D_{LL} - D_{NN})\left[\frac{4r_j}{r^2} + r_i r_j \frac{r_i}{r}\left(-\frac{2}{r^3}\right)\right]$$

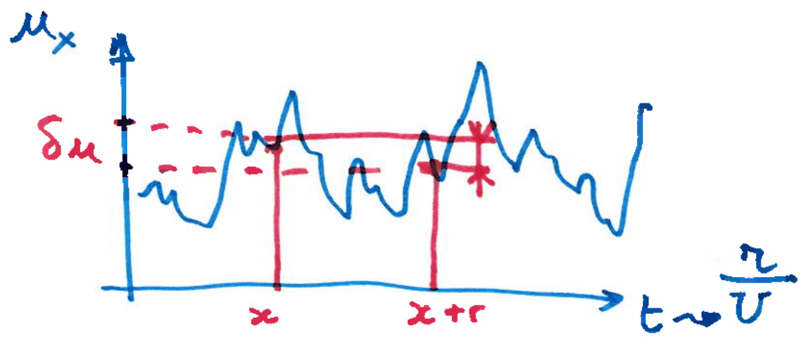
$$= \frac{r_j}{r}\left\{\cancel{\frac{\partial D_{NN}}{\partial r}} + \frac{\partial}{\partial r}\left(D_{LL} - \cancel{D_{NN}}\right) + (D_{LL} - D_{NN})\frac{2}{r}\right\}.$$

$$= \frac{r_j}{r}\left\{\frac{\partial D_{LL}}{\partial r} + (D_{LL} - D_{NN})\frac{2}{r}\right\} = 0$$

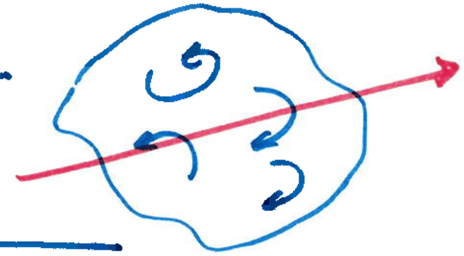
$$\boxed{D_{NN} = D_{LL} + \frac{1}{2} r \frac{\partial}{\partial r} D_{LL}}$$

Pour des échelles $\underline{\eta \ll r \ll L}$

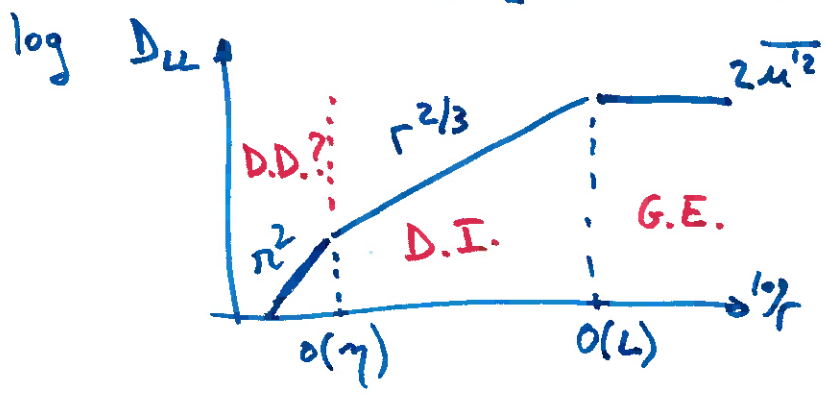
$$D_{LL}(r) = f(r,\varepsilon) = C_2 (\varepsilon r)^{2/3} \quad , \text{ où } \boxed{C_2 = O(1)}$$



Hypothèse de "Turbulence gelée".



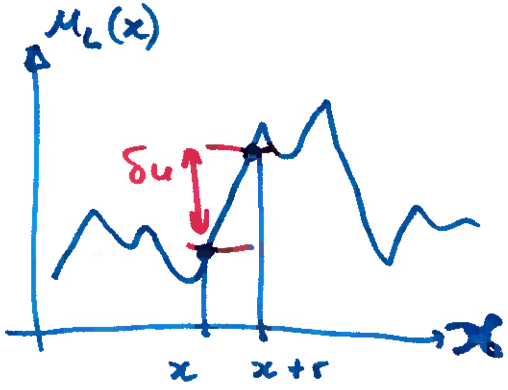
$$D_{LL} = \overline{[u_L(x+r) - u_L(x)]^2}$$



$$\boxed{C_2 \simeq 2{,}0 \pm 0{,}2}$$

$r \lesssim 20\,\eta \rightarrow$ Dissipatif.

$r \gg L$,
$$D_{LL} = \overline{u_L^2(x+r)} + \overline{u_L^2(x)} - \cancel{2\overline{u_L(x)\,u_L(x+r)}}$$
$$\rightarrow 2\,\overline{u'^2}$$



$$u_i(\vec{x}_0+\vec{r}) \simeq u_i(\vec{x}_0) + r_j \left.\frac{\partial u_i}{\partial x_j}\right|_{\vec{x}_0} + o(r^2)$$

$$u_i(\vec{x}_0+\vec{r}) - \vec{u}_i(x_0) = r_j \frac{\partial v_i}{\partial x_j}.$$

$$\overline{\left[u_L(x+r) - u_L(x)\right]^2} = r^2 \overline{\left(\frac{\partial u_x}{\partial x}\right)^2}$$

$D_{LL}(r) \propto r^2$ pour $r \leqslant 20\eta$.

$$\rightarrow \left[\, D_{NN} = D_{LL} + \frac{1}{2} r \frac{\partial}{\partial r} D_{LL}. \right.$$

$r \leqslant 20\eta$. $\quad D_{NN} = r^2 \overline{\left(\frac{\partial u_y}{\partial x}\right)^2} = r^2 \overline{\left(\frac{\partial u_x}{\partial y}\right)^2} = r^2 \overline{\left(\frac{\partial u_z}{\partial y}\right)^2}$

$$D_{NN} = r^2 \overline{\left(\frac{\partial u_x}{\partial y}\right)^2} = r^2 \overline{\left(\frac{\partial u_x}{\partial x}\right)^2} + r^2 \overline{\left(\frac{\partial u_x}{\partial x}\right)^2}$$

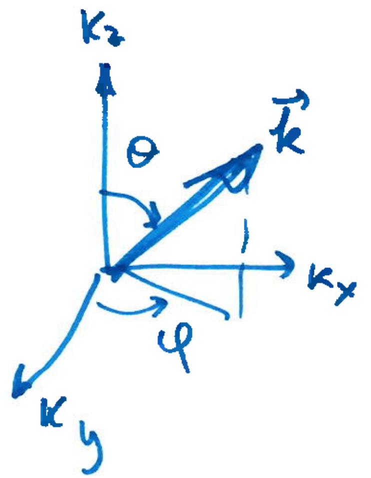
d'où $\boxed{\overline{\left(\frac{\partial u_x}{\partial y}\right)^2} = 2\, \overline{\left(\frac{\partial u_x}{\partial x}\right)^2}}$

$$\varepsilon = 2\nu \overline{s_{ij}' s_{ij}'} = \underbrace{\nu \overline{\left(\frac{\partial u_i}{\partial x_j}\right)^2}}_{\text{Pseudo-Dissip}^{\circ}.} = \nu \Bigg[\underbrace{\overline{\left(\frac{\partial v_x}{\partial x}\right)^2} + \overline{\left(\frac{\partial v_y}{\partial y}\right)^2} + \overline{\left(\frac{\partial v_z}{\partial z}\right)^2} + \overline{\left(\frac{\partial v_x}{\partial y}\right)^2} + \text{etc}}_{6 \text{ termes.}} \Bigg]$$

$$= \nu \overline{\left(\frac{\partial v_x}{\partial x}\right)^2} \left[3 + 2 \times 6 \right] = 15 \nu \overline{\left(\frac{\partial v_x}{\partial x}\right)^2}$$

c) Spectre d'Energie.

$$u_i(\vec{x}) \longrightarrow \boxed{\hat{u}_i(\vec{k}) = \frac{1}{(2\pi)^3} \iiint e^{i \vec{k} \cdot \vec{x}} \, u_i(\vec{x}) \, d^3\vec{x}} \quad m^4 s^{-1}$$



Spectre : $E_{3D}(\vec{k}) = \cancel{\quad} \frac{1}{2} |\hat{u}_i(\vec{k})|^2 \quad m^8 s^{-2}$

Spectre 1D. $E_{1D}(k) = \iint E_{3D}(\vec{k}) \, k^2 \sin\theta \, d\theta \, d\varphi$

$$\boxed{E_{1D}(k) = 4\pi k^2 E_{3D}(\vec{k})} \quad m^6 s^{-2}$$

Propriété :

$$E_{1D}(k) \underset{\sim}{\simeq} « \int_0^\infty D_{LL}(r)\, e^{ikr}\, dr »$$

$$\simeq C_2 \int (\varepsilon r)^{2/3} e^{ikr}\, dr \qquad (\eta \ll r \ll L).$$

$1/L \ll k \ll 1/\eta$.

$y = kr$

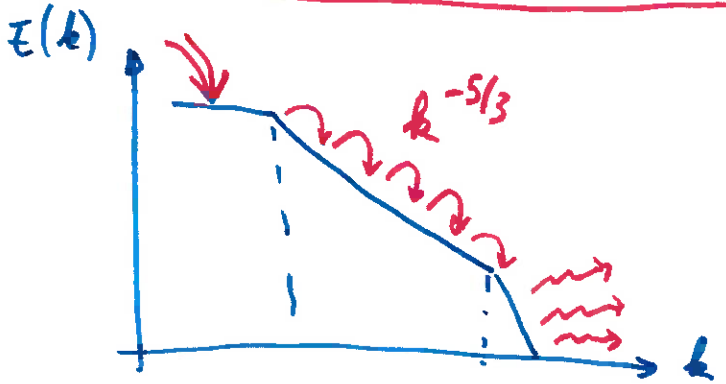
$$\simeq C_2 \varepsilon^{2/3} \int \left(\frac{y}{k}\right)^{2/3} e^{iy} \frac{dy}{k}$$

$$\simeq C_2 \varepsilon^{2/3} k^{-5/3} \underbrace{\int y^{2/3} e^{iy}\, dy}_{O(1)}.$$

$$\boxed{E_{1D}(k) = C_K\, \varepsilon^{2/3} k^{-5/3}}$$

Spectre d'energie de Kolmogorov.

Loi des $-5/3$.



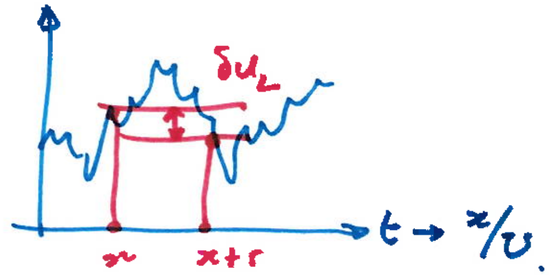
D) Fonction de structure d'ordre 3.

$$D_{ijk}(\cancel{\vec{x}}, \vec{r}, \cancel{t}) = \overline{[u_i' - u_i][u_j' - u_j][u_k' - u_k]}.$$

$\hookrightarrow |\vec{r}| = r$

$u_i' = u_i(\vec{x}+\vec{r})$ $\quad$ $u_i = u_i(\vec{x})$

$$\ldots \quad D_{LLL}(r) = \overline{[u_L(\vec{x}+\vec{r}) - u_L(\vec{x})]^3}.$$



$$\delta u_L = u_L' - u_L$$

$$\begin{cases} \bullet\ \overline{\delta u_L} = 0 \\ \bullet\ \overline{\delta u_L^2} = c_2 (\varepsilon r)^{2/3}. \\ \bullet\ \overline{\delta u_L^3} \neq 0. \end{cases}$$

Relation de Kármán–Howarth.

$$[NS(u_i) - NS(u_i')] \cdot (u_j' - u_j) \quad + \text{sym}.$$

$$\ldots \quad \boxed{\frac{\partial}{\partial t} D_{ij} = f(D_{ijk}) + \text{visqueux.}}$$

$$\ldots \quad \boxed{\frac{D_{LLL}}{r} - \frac{6\nu}{r} \frac{\partial D_{LL}}{\partial r} = -\frac{4}{5}\epsilon}$$

Dom. Dissipatif, $r \to 0$.

$$\frac{6\nu}{r}\frac{\partial D_{LL}}{\partial r} = \frac{4}{5}\epsilon$$

$$\frac{\partial D_{LL}}{\partial r} = \frac{2}{15}\frac{\epsilon}{\nu} r.$$

$$\boxed{D_{LL}(r) = \frac{\varepsilon}{15\nu} r^2}$$

(NB: $\varepsilon = 15\nu \overline{\left(\frac{\partial u}{\partial x}\right)^2}$, et $D_{LL} = r^2 \overline{\left(\frac{\partial u_x}{\partial x}\right)^2}$).

Dom. Inertiel, $\eta \ll r \ll L$

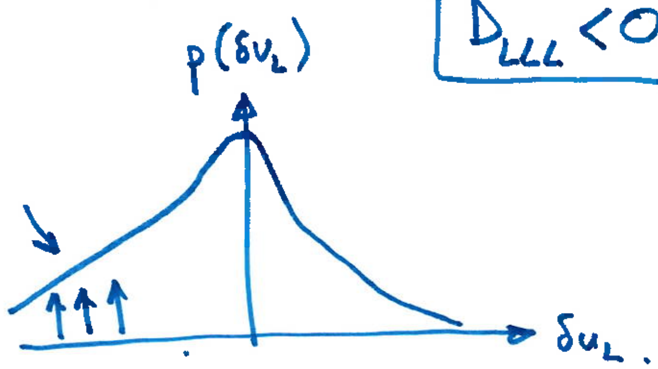
$$\boxed{D_{LLL} = -\frac{4}{5}\epsilon r}$$ Loi des 4/5 de Kolmogorov

$m^3 s^{-3}$

$$D_{LL} = C_2 (\varepsilon r)^{2/3}$$

?

$\boxed{D_{LLL} < 0}$ $\Longleftrightarrow$ Cascade d'Energie Grandes $\rightarrow$ Petites échelles.



pdf: probability density function.

$$\begin{cases} D_{LLL} = -\frac{4}{5}\epsilon r \,, & \eta \ll r \ll L. \\ D_{LLL} \propto - r^3 & \eta \lesssim 20\eta. \end{cases}$$

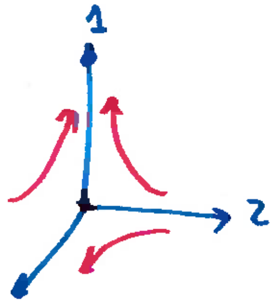
$$D_{LLL} \simeq r^3 \overline{\left(\frac{\partial u_x}{\partial x}\right)^3} \Rightarrow \boxed{\overline{\left(\frac{\partial u_x}{\partial x}\right)^3} < 0.}$$

$$\frac{1}{2}(\partial_i v_j + \partial_j v_i) = S_{ij}\Big|_{123} = \begin{pmatrix} \gamma_1 & & \\ & \gamma_2 & \\ & & \gamma_3 \end{pmatrix}$$

où incompr. $\Rightarrow \gamma_1 + \gamma_2 + \gamma_3 = 0.$

$\gamma_1 > 0 \quad \gamma_3 < 0.$

$\gamma_2 ?$



$$= \begin{pmatrix} \partial v_x/\partial x & 0 & 0 \\ 0 & \partial v_y/\partial_y & 0 \\ 0 & 0 & \partial v_z/\partial z \end{pmatrix}$$

$$\overline{\left(\frac{\partial u_x}{\partial x}\right)^3} \simeq \frac{1}{3}\left(\gamma_1^3 + \gamma_2^3 + \gamma_3^3\right)$$

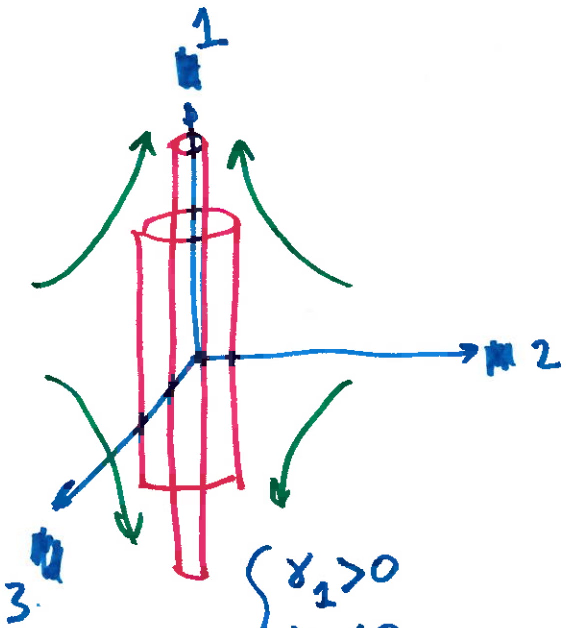
$$= \frac{1}{3}\left(\gamma_1^3 + \gamma_2^3 + (-\gamma_1 - \gamma_2)^3\right)$$

$$= \frac{1}{3}\left(\gamma_1^3 + \gamma_2^3 - \gamma_1^3 - 3\gamma_1\gamma_2^2 - 3\gamma_1^2\gamma_2 - \gamma_2^3\right)$$

$$= \frac{1}{3}\left(-3\gamma_1\gamma_2^2 - 3\gamma_1^2\gamma_2\right)$$

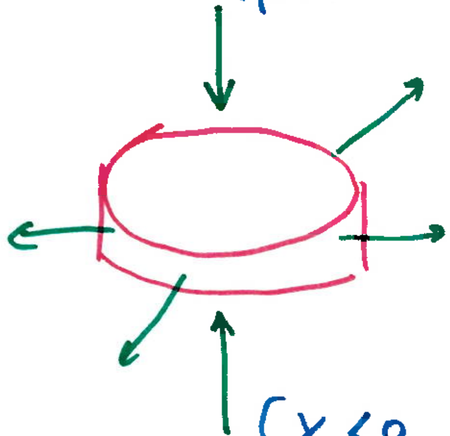
$$= -\gamma_1\gamma_2(\gamma_2+\gamma_1) = \gamma_1\gamma_2\gamma_3 . < 0$$

$\gamma_1 > 0$ $\quad$ $\boxed{\gamma_2 > 0}$!!! $\quad$ $\gamma_3 < 0$



$$\begin{cases} \gamma_1 > 0 \\ \gamma_2 < 0 \\ \gamma_3 < 0 \end{cases}$$

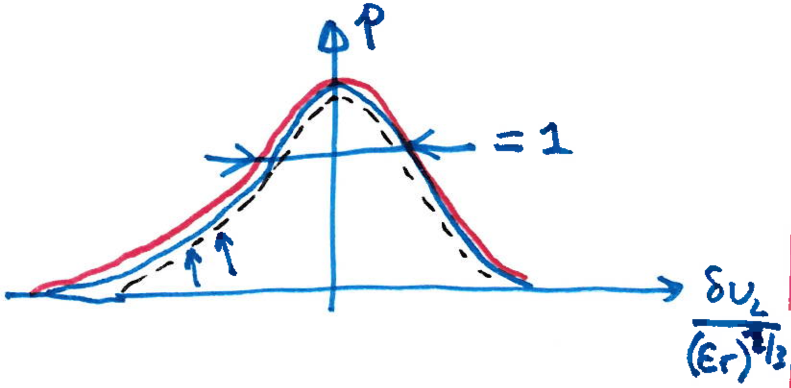
$\gamma_1 < 0$.

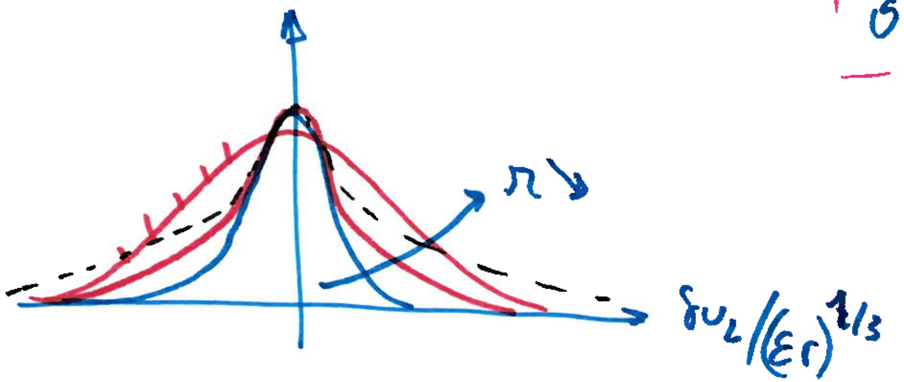


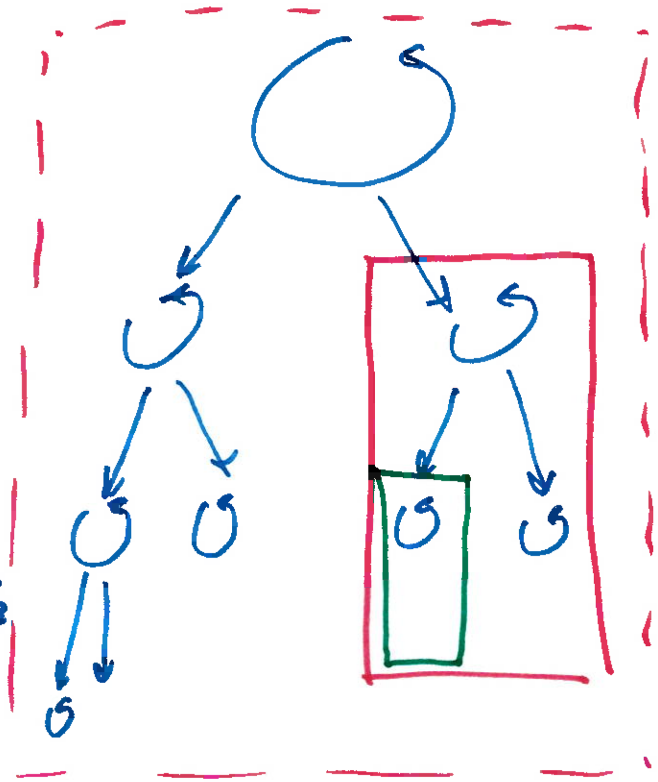
$$\begin{cases} \gamma_1 < 0 \\ \gamma_2 > 0 \\ \gamma_3 > 0. \end{cases}$$

V. Notions sur l'intermittence.

$$D_{LL} = C_2 (\varepsilon r)^{2/3} = \overline{[\delta u_L]^2}$$







Si Turb. est Auto-similaire.

$P\left(\frac{\delta u_L(r)}{(\varepsilon r)^{1/3}}\right)$ est indépendante de r.

$$\boxed{\overline{\delta u_L^n} \simeq (\varepsilon r)^{n/3}}$$